TOPOLOGY COURSE 2024-25
EXERCISE SOLUTIONS

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Contents

Terms and conditons

You are encouraged to use this document responsibly: you should always try the hardest possible to solve the exercises on your own, and only then check your answers. Also, after you read a solution, always take a moment to meditate about what you learnt from it!
If you find any errors, or you want to submit a more elegant solution, please write an e-mail to *gm2070@hw.ac.uk*.

Confusing notation

$\mathbb{N}$ will denote the set of natural numbers, *including zero* (the absence of something is as natural as the presence of something). Whenever I want to exclude zero I will write $\mathbb{N}_{>0}$.

Chapter 2: Set theoretic revision

Exercise 1. We shall prove that, given a map $f\colon X \to Y$ between sets and any $A, B \subset Y$, the following relation holds:

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

To prove that two sets are equal, we must show that every element belonging to one also belongs to the other, and viceversa.[1]
So first we show that every element x which belongs to $f^{-1}(A \cap B)$ must also belong to $f^{-1}(A) \cap f^{-1}(B)$. Now, $x \in f^{-1}(A \cap B)$ means that $f(x) \in A \cap B$. As $A \cap B \subseteq A$, we have that $f(x) \in A$, which in turn means that $x \in f^{-1}(A)$. If we replace A with B in the above argument[2] we get that $x \in f^{-1}(B)$ as well. As x

[1] This is what is called the *extensionality axiom*.
[2] We can because A and B play symmetrical roles in this proof!

belongs to both $f^{-1}(A)$ and $f^{-1}(B)$, it must lie in the intersection of these two sets, so $x \in f^{-1}(A) \cap f^{-1}(B)$.
Conversely, pick any element $x \in f^{-1}(A) \cap f^{-1}(B)$, and we have to show that $x \in f^{-1}(A \cap B)$. Notice that $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A)$, so $x \in f^{-1}(A)$, which means that $f(x) \in A$. The same argument, with A replaced by B, yields that $f(x) \in B$. Thus $f(x)$ belongs to $A \cap B$, as it belongs to both A and B, and in turn this means that $x \in f^{-1}(A \cap B)$, as required.
The proof of the second equality is very similar.

Exercise 2. As in the previous exercise, we show that the two sides of the equality have the same elements. To this extent, we shall prove that, given any element $x \in X$, then $x \in f^{-1}(Y - A)$ if and only if $x \in X - f^{-1}(A)$.[3]. Indeed, $x \in f^{-1}(Y - A)$ if and only if $f(x) \in Y - A$, as this is the definition of preimage of $Y - A$. In turn, $f(x) \in Y - A$ is equivalent to saying that $f(x) \notin A$, and in turn $f(x) \notin A$ if and only if $x \notin f^{-1}(A)$. Finally, we can rewrite the last statement as $x \in X - f^{-1}(A)$, as required.

Chapter 3: Topologies and continuity

Exercise 3. Recall that a subset O of $\mathbb{R}$ is open if, for every $x \in O$, there exists an open interval (a, b) such that $x \in (a, b)$ and $(a, b) \subset O$.
Firstly, $\mathbb{R}$ is open, as every point $x \in \mathbb{R}$ belongs to some open interval (say for example $(x - 1, x + 1)$). Furthermore, every open interval (a, b) is tautologically an open subset of $\mathbb{R}$.
Now let F be a finite subset of $\mathbb{R}$, and enumerate its elements $x_1, \dots, x_k$ for some $k \in \mathbb{N}$. For every $x \in \mathbb{R} - F$, there exists an interval $(x - \varepsilon, x + \varepsilon)$ which is small enough to contain x but not any element of F: for example, one can take

$$\varepsilon = 1/2 \min_{i=1,\dots,k} |x - x_i|,$$

where $|x - x_i|$ is the distance between x and the ith point of F. Then $(x - \varepsilon, x + \varepsilon) \subset \mathbb{R} - F$. This proves that $\mathbb{R} - F$ is open.
Finally, let $\varnothing$ be the empty set. As no element $x \in R$ belongs to $\varnothing$ (which is, indeed, empty), it is true that every element of the empty set (that is, no element) belongs to an open interval contained in the empty set, because *there are no elements to check*! [4]

Exercise 4. Let $a \leqslant b$ be elements of $\mathbb{R}$. The closed interval $[a, b]$ is *not* open, and to show this it is enough to exhibit an element p of $[a, b]$ which does not belong to any open interval contained in $[a, b]$. We claim that we can choose $p = b$. Indeed, if an open interval (x, y) contains b then $b < y$, so (x, y) also contains $b + \varepsilon$ for some small enough $\varepsilon > 0$. In turn, this means that (x, y) cannot be contained in $[a, b]$, as it contains the point $b + \varepsilon$ which is strictly greater than b.
Moreover, let $F = \{x_1, \dots, x_k\}$ be a finite, non-empty set, for some $k \in \mathbb{N}_{>0}$. Then any open interval (x, y) containing x_1 must also contain $x + \varepsilon$ for some small ε, and we can choose it small enough that $x + \varepsilon$ is not any of the other points $x_2, \dots, x_k$.

[3]This could be done also in the previous exercise: try it!

[4]Think of it this way: the definition of some set O being open can be rephrased by saying "if an element $x \in \mathbb{R}$ belongs to O, then...". This is a logical implication, and if $O = \varnothing$ then the hypothesis of the implication is false, as no element of $\mathbb{R}$ can belong to $\varnothing$. Then an implication with false premise is always true, logically speaking!

This means that no open interval can contain x_1 and be contained in F, so F is not open.

Exercise 5. We have to check the following three requirements:

(1) $\varnothing$ and $\mathbb{R}$ are both open. This was part of Exercise 3.
(2) If $\{A_i\}_{i\in I}$ is any collection of open subsets, where i varies in some index set I, then the union $A = \bigcup_{i\in I} A_i$ is open as well. Indeed, every $x \in A$ must belong to some A_i, and as A_i is open there must be an open interval $(a, b) \subseteq A_i$ containing x. In particular, (a, b) is also contained in A. This proves that every point x of A belongs to some open interval contained in A, that is, A is open.
(3) If $\{A_i\}_{i=1,\dots,k}$ is any *finite* collection of open subsets, then the intersection $A' = \bigcap_{i=1}^k A_i$ is open as well. Indeed, if $x \in A'$ then $x \in A_i$ for every i, so we can find an open interval $(a_i, b_i) \subseteq A_i$ containing x. In other words, $a_i < x < b_i$ for every i, meaning that $x \in (\max_{i=1,\dots,k} a_i, \min_{i=1,\dots,k} b_i)$. As the latter interval is contained in every (a_i, b_i), we see that
$$(\max_{i=1,\dots,k} a_i, \min_{i=1,\dots,k} b_i) \subseteq A'.$$
This proves that A' is open.

Exercise 6. We have to check the following three requirements:

(1) $\varnothing$ and $\mathbb{R}$ both belong to $\mathcal{O}_1$. This is true as $\varnothing \in \mathcal{O}_1$ by construction, while $\mathbb{R} = \mathbb{R} - \varnothing$ is obtained from $\mathbb{R}$ after removing zero (hence finitely many) points.
(2) If $\{A_i\}_{i\in I} \subseteq \mathcal{O}_1$ then $A = \bigcup_{i\in I} A_i$ also belongs to $\mathcal{O}_1$. Indeed each (non-empty) A_i can be written as $\mathbb{R}-F_i$ for some finite set F_i. So $A = \mathbb{R}-\bigcap_{i\in I} F_i$, and a intersection of finite sets is finite.
(3) If $\{A_i\}_{i=1,\dots,k} \subseteq \mathcal{O}_1$ for some $k \in \mathbb{N}$, then the intersection $A' = \bigcap_{i=1}^k A_i$ belongs to $\mathcal{O}_1$ as well. Indeed, if one of the A_i is empty then clearly $A' = \varnothing$. Otherwise each A_i can be written as $\mathbb{R} - F_i$ for some finite subset F_i; therefore $A' = \mathbb{R} - \bigcup_{i=1}^k F_i$ is in $\mathcal{O}_1$ as we are removing a finite union of finite sets, which is again finite.

Exercise 7 (hint). This is proven as Exercise 5: just replace every open interval (a, b) with $[a, b)$.

Exercise 8 (hint). It is enough to find a subset which is open in $\mathcal{O}_{std}$ but not in $\mathcal{O}_1$. For example, $(0, 1)$ is open in the standard topology (as it is an open interval), but is not open in $\mathcal{O}_1$ as it cannot be espressed as $\mathbb{R}$ minus a finite set.

Exercise 9 (hint).

(1) (a, b) is open, by how the standard topology is defined. However it is not closed, because its complement $(-\infty, a] \cup [b, +\infty)$ is *not* open (to see this, one can argue exactly as in Exercise 4).
(2) $[a, b]$ is closed, because its complement $(-\infty, a) \cup (b, +\infty)$ is open since it is a union of open intervals. However, $[a, b]$ is not open, as shown in Exercise 4.
(3) $[a, b)$ is not open, which again can be seen by repeating the proof of Exercise 4. For the same reason, its complement $(-\infty, a) \cup [b, +\infty)$ is not open, which means that $[a, b)$ is not closed.

(4) $\mathbb{R}$ is open, as the whole space always belong to a topology. $\mathbb{R}$ is also closed, as its complement is $\varnothing$ which is also open.

Exercise 10.

(1) As the whole space X belongs the topology, it is a union $\bigcup i \in IB_i$ of elements of the basis. Hence every $x \in X$ must belong to some $B_i \in \mathcal{B}$.
(2) As $B_1, B_2 \in \mathcal{B} \subseteq \mathcal{O}$, both B_1 and B_2 are open, so their intersection is open as well. Therefore $B_1 \cap B_2$ is a union of elements of $\mathcal{B}$, and therefore every $x \in B_1 \cap B_2$ belongs to some basis element $B_3 \subseteq B_1 \cap B_2$.

Exercise 11. By Proposition 3.10, it is enough to check that the given basis $\mathcal{B} = \{O \times V : O \in \mathcal{O}, V \in \mathcal{V}\}$ satisfies the two properties from Exercise 10.

(1) Every $(x, y) \in X \times Y$ belongs to $X \times Y$, which is in the basis as $X \in \mathcal{O}$ and $Y \in \mathcal{V}$.
(2) Given two elements $O_1 \times V_1$, $O_2 \times V_2$ of the basis, it is easy to see that their intersection is $(O_1 \cap O_2) \times (V_1 \cap V_2)$, which is again a product of open sets (as any finite intersections of open subsets is open). Therefore $(O_1 \cap O_2) \times (V_1 \cap V_2) \in \mathcal{B}$.

Exercise 12. Let $O \in \mathcal{O}_1$, and we want to show that it also belongs to $\mathcal{O}_2$. By definition of the basis $\mathcal{B}_1$, we have that O is some union $\bigcup_{i\in I} B_i$, where every $B_i \in \mathcal{B}_1$. Moreover for every $i \in I$ and every $x \in B_i$ there exists $B'(x, i) \in \mathcal{B}_2$ such that $x \in B'(x, i) \subseteq B_i$. Hence

$$O = \bigcup_{i\in I} B_i = \bigcup_{i\in I} \bigcup_{x\in B_i} B'(x, i).$$

In other words, O is a union of elements of the basis $\mathcal{B}_2$, and therefore belongs to $\mathcal{O}_2$, as required.

Exercise 13 (hint). Let $d(\cdot, \cdot)$ denote the Euclidean distance on $\mathbb{R}^n$. In order to show that the metric topology coincides with the standard topology, one has to show that any Euclidean ball $B(x, r) = \{p \in \mathbb{R}^n : d(x, p) < r\}$ is a union of open rectangles $(a_1, b_1) \times \ldots, \times (a_, b_n)$, and viceversa every rectangle is a union of balls. It isn't hard to imagine a ball made of tiny little rectangles. . .

Exercise 14. We need to check the three properties of a topology:

(1) A is in $\mathcal{O}_A$, as $A = A \cap X$ and X is open in any topology on X.
(2) Any union of elements of $\mathcal{O}_A$ is of the form $\bigcup_{i\in I}(A \cap O_i) = A \cap (\bigcup_{i\in I} O_i)$, where every $O_i \in \mathcal{O}$. As $\bigcup_{i\in I} O_i \in \mathcal{O}$, we get that the union is in $\mathcal{O}_A$ as well.
(3) Any finite intersection of elements of $\mathcal{O}_A$ is of the form $\bigcap_{i=1}^k (A \cap O_i) = A \cap \left(\bigcap_{i=1}^k O_i\right)$, where every $O_i \in \mathcal{O}$. As $\bigcap_{i=1}^k O_i \in \mathcal{O}$, we get that the intersection is in $\mathcal{O}_A$ as well.

Exercise 15 (partial solution). We shall prove that the composition of two continuous maps $f : X \to Y$ and $g : Y \to Z$ is continuous. Let $A \subseteq Z$ be an open subset, and we want to show that $(g \circ f)^{-1}(A)$ is open. By definition,

$$(g \circ f)^{-1}(A) = f^{-1}\left(g^{-1}(A)\right).$$

As g is continuous, the preimage $g^{-1}(A)$ of the open set A is open in Y. But then, using that f is continuous as well, we get that the preimage $f^{-1}\left(g^{-1}(A)\right)$ is open in X, as required.
The other two parts of the exercise are dealt with similarly (try them!)

Exercise 16. The identity map $id\colon (X, \mathcal{O}_{disc}) \to (X, \mathcal{O}_{triv})$ is clearly bijective: every element $x \in X$ is mapped to x, so the inverse of the identity map is the identity map itself. Moreover, this map is continuous: the only elements of $\mathcal{O}_{triv}$ are $\varnothing$ and X, and their preimages under the identity map (which are again $\varnothing$ and X) belong to the discrete topology $\mathcal{O}_{disc}$, and actually to any topology on X.
However, to prove that id is not a homeomorphism, we must prove that the inverse map $id\colon (X, \mathcal{O}_{triv}) \to (X, \mathcal{O}_{disc})$ is *not* continuous. To see this, let $\{x\}$ be the subset containing a single element $x \in X$. Notice that $\{x\}$ is open in the discrete topology, which contains every subset of X. However, $\{x\}$ does not coincide with neither the empty set $\varnothing$ nor the whole space X, because we are assuming that X has at least two points. Therefore, $id^{-1}(\{x\}) = \{x\}$ is *not* an element of the trivial topology. This means that $id\colon (X, \mathcal{O}_{triv}) \to (X, \mathcal{O}_{disc})$ is discontinuous, as we found a subset of X which is open in the discrete topology, but whose preimage under id is not open in the trivial topology.

Exercise 17. The proof is by contradiction: we assume that the thesis is true, and we deduce an impossible statement.
So suppose that $\mathcal{O}_1$ is first-countable. This means that, if we fix $x = 0$, there exists a countable family $A_n = \mathbb{R} - F_n$ of elements of $\mathcal{O}_1$, such that:

- each A_n contains 0;
- If $O \in \mathcal{O}_1$ contains 0 then it must also contain A_k for some $k \in \mathbb{N}$.

We claim that we can find an $O \in \mathcal{O}_1$ which does not satisfy the second statement. Indeed, let $F = \bigcup_{n\in\mathbb{N}} F_n$ be the union of the finite sets that define the A_ns. This is a countable union of finite sets, and is therefore countable.[5] In particular, $F \cup \{0\}$ is still countable.[6] As $\mathbb{R}$ is uncountable, it cannot coincide with $F \cup \{0\}$, so there must be some element $y \in \mathbb{R} - (F \cup \{0\})$.
Then let $O = \mathbb{R} - \{y\}$, which belongs to the topology $\mathcal{O}_1$. We claim that, for every $n \in \mathbb{N}$, $A_n \nsubseteq O$. Indeed, the element y belongs to A_n, as $A_n = \mathbb{R} - F_n$ and by construction F_n does not contain y; however $y \notin O$ by our choice of O, meaning that A_n cannot be a subset of O.
In other words, we found an element of the topology which contains 0 but does not contain any of the A_ns, against the fact that the A_n form a neighbourhood basis. □

Exercise 18. Let $A_n = O_1 \cap \ldots \cap O_n$. To show that A_n is a neighbourhood basis for x we must check the following facts:

- Every A_n contains x. This is clearly true, as every O_k contains x and therefore any intersection of O_k contains x.
- If $A \in \mathcal{O}$ is open and contains x, then it contains some A_n. Indeed, by definition of the neighbourhood basis $\{O_n\}$, there exists $n \in \mathbb{N}$ such that $O_n \subseteq A$, and now it suffices to notice that $A_n \subseteq O_n$ by construction.

[5]Try to prove this yourself, Otherwise, check `https://math.stackexchange.com/questions/603456/prove-that-the-union-of-countably-many-countable-sets-is-countable`.

[6]This is the famous children game ”Infinity plus one”...

Exercise 19. We first point out the following facts:

(1) In any topological space X, if $C \subseteq X$ is closed then, for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq C$ and any limit $\bar{x}$ for $(x_n)_{n\in\mathbb{N}}$, then $\bar{x} \in C$ (this is the solution of the question from Remark 3.21; try to prove it yourself!)
(2) Let X be a *first-countable* topological space X, and let $C \subseteq X$ be such that, for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq C$ and any limit $\bar{x}$ for $(x_n)_{n\in\mathbb{N}}$, the limit belongs to C. Then C is closed (this is Lemma 3.20).

Having this in mind, an inspection of the proof of Lemma 3.22 reveals that the only spot where we really need one of the spaces to be first-countable is where we say that, to prove that $f^{-1}(C)$ is closed, it is enough to prove that, given any sequence $(x_n)_{n\in\mathbb{N}} \subseteq f^{-1}(C)$, any limit $\bar{x}$ for $(x_n)_{n\in\mathbb{N}}$ belongs to $f^{-1}(C)$. This is point (2) from above, and is only using the fact that X is first-countable.
Another way to realise that the requirement on X is the only that we really need is to check that the Lemma is false if X is not first-countable, even assuming that Y is. Indeed, let $X = \mathbb{R}$ with the *cocountable topology* (a subset A is open if and only either A is empty, or $A = \mathbb{R} - T$, where T is countable), and let $Y = \mathbb{R}$ with the Euclidean topology. It is easy to see that Y is first-countable, while X is not (for the latter, the proof is very similar to that of Exercise 17). Now consider the identity map $id\colon X \to Y$.

- First notice that id is not continuous. Indeed, the interval $(0, 1)$ is open in the usual topology, but not in the cocountable one.
- However, id maps converging subsequences to converging subsequences. Indeed, let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a converging subsequence, and let $\bar{x}$ be any of its limits. Consider the set $A = \mathbb{R} - \{x_n\}_{n\in\mathbb{N}} \cup \{\bar{x}\}$. This set is open in the cocountable topology, as we are removing countably many points from $\mathbb{R}$. As x_n converges to $\bar{x}$, there exists n_0 such that $x_n \in A$ for every $n \geqslant n_0$. By how we defined A, this means that $x_n = \bar{x}$ for every $n \geqslant n_0$, that is, the sequence is *constant* from a certain point. Then, if we take the image under the identity map, we get that x_n still converges to $\bar{x}$ in Y, as it eventually stabilises. This proves that the identity map sends every subsequence which converges in X to a subsequence which converges in Y.

Chapter 4: Connected spaces

Exercise 20. Clearly $f|_{X-\{x\}}$ is bijective. Moreover, f is continuous. Indeed, let $A \subseteq Y - \{f(x)\}$ be open in the subspace topology, meaning that there is a subset $A' \subseteq Y$ which is open in Y and such that $A' - \{f(x)\} = A$. By continuity of f, $f^{-1}(A')$ is open, and notice that $f^{-1}(A') - \{x\} = f^{-1}(A)$ as f is bijective. Hence $f^{-1}(A)$ is open in the subspace topology for $X - \{x\}$, as required. Finally, if in the above argument we replace f by its inverse $f^{-1}\colon Y \to X$, we get that the inverse of $f|_{X-\{x\}}$ is also continuous. Therefore $f|_{X-\{x\}}$ is a homeomorphism.

Exercise 21. 1. $\to$ 2. If $X = A \sqcup B$ and both A and B are closed, then $X - A = B$ and $X - B = A$ must be open as the complement of a closed set is open.
2. $\to$ 3. If A is open and $B = X - A$ is open, then A is also closed. Moreover, as A and B are non-trivial, A is neither empty nor the whole X.
3. $\to$ 1. If A is non-empty, not X, and both closed and open, then $B \coloneqq X - A$ is non-empty and closed.

Exercise 22. The subsets $(-\infty, 0)$ and $(0, +\infty)$ are non-empty, open subsets of $\mathbb{R} - \{0\}$ whose union is the whole space. Therefore $\mathbb{R} - \{0\}$ is disconnected.

Exercise 23. Consider the subsets $A = (-\infty, \pi) \cap \mathbb{Q}$ and $B = (\pi, +\infty) \cap \mathbb{Q}$. They are open in $\mathbb{Q}$, as they are obtained by intersecting $\mathbb{Q}$ with two open sets of $\mathbb{R}$, and their union is the whole $\mathbb{Q}$ as $\pi \notin \mathbb{Q}$.

Exercise 24 (self-explanatory). There aren't that many non-trivial subsets of $\{p\}$...

Exercise 25 (updated 10/10/2024). Let $f \colon X \to Y$ be a continuous, surjective map. We shall prove the contrapositive of the statement: if Y is disconnected then X must be disconnected as well. Indeed, Y being disconnected means that there exist two non-empty open subsets A and B of Y such that $A \sqcup B = Y$. Now, their preimages under f are open subsets of X, as f is continuous, and $f^{-1}(A) \cap f^{-1}(B) = \varnothing$ as the targets A and B are disjoint. Furthermore, as f is surjective, there exists $x, x' \in X$ such that $f(x) \in A$ and $f(y) \in B$, so $f^{-1}(A)$ and $f^{-1}(B)$ are both non-empty. Then $X = f^{-1}(A) \sqcup f^{-1}(B)$ is a disjoint union of non-empty open subsets.
In particular, if X and Y are homeomorphic, there exists a continuous map $f \colon X \to Y$ whose inverse is continuous, so X is connected if and only if Y is.

Exercise 26 (hint). Simply take the negation of Item 3. of the definition of a disconnected space.

Exercise 27 (hint). A is closed, so its complement $[a, b] - A$ is open (which is what we used in the proof).

Exercise 28. Suppose by contradiction that X is not connected. Hence we can find two non-empty open subsets O, P such that $X = O \sqcup P$. As $X = \bigcup_{n \in \mathbb{N}} A_n$ and X strictly contains both O and P, there must be some $n \in \mathbb{N}$ such that A_n intersects non-trivially both O and P (otherwise all A_n would line in, say, O, and therefore X would be contained in O). Hence, by definition of the subspace topology on A_n, we have that $A_n \cap O$ and $A_n \cap P$ are non-empty open subsets of A_n, contradicting the fact that A_n was connected. $\square$

Exercise 29. Given any $n \in \mathbb{N} - \{0\}$ and any two points $x, y \in \mathbb{R}^n$, consider the linear map $f \colon [0, 1] \to \mathbb{R}^n$ such that, for every $t \in [0, 1]$,

$$f(t) = ty + (1 - t)x,$$

where we see x and y as vectors in $\mathbb{R}^n$ and take a linear combination depending on t. What is really going on is that the image of f is the line segment between x and y. One can easily check that $f(0) = x$, $f(1) = y$, and f is continuous.

Exercise 30 (hint). For simplicity, we assume $n = 2$, as the following argument easily generalises to higher dimensions. Given any two points $x, y \in R^2 - \{0\}$ we shall describe a path from x to y, without giving its explicit function. First, consider the circle C centred at 0 and passing through x. Let $x' \in C$ be such that 0, x', and y are aligned, and 0 does not lie between x' and y. Then the path from x to y is the circle arc from x to x', followed the line segment from x' to y (notice that, by our choice of x', this segment does not contain 0). We thus get a path from x to y which is totally contained in $R^2 - \{0\}$.

Exercise 31. A path from x to x always exists: it's the constant path $f\colon [0,1] \to X$ such that $f(t) = x$ for every $t \in [0,1]$. Therefore x lies in its connected component $P(x)$.

Exercise 32. We shall prove the exercise in the case when $X = A \cup B$, for A and B open (the case in which they are both closed follows analogously). Let $O \subseteq Y$ be open, and we want to show that $f^{-1}(O)$ is open. Notice that, as $f|_A\colon A \to Y$ is continuous, we have that $f|_A^{-1}(O) = f^{-1}(O) \cap A$ is open, and similarly $f^{-1}(O) \cap B$ is open. Then

$$f^{-1}(O) = f^{-1}(O) \cap X = f^{-1}(O) \cap (A \cup B) = (f^{-1}(O) \cap A) \cup (f^{-1}(O) \cap B).$$

Therefore $f^{-1}(O)$ is a union of two open sets, and is therefore open.

Exercise 33. 1. Given any two $y, z \in P(x)$, there exist a path from y to x and a path from x to z, so their concatenation is a path from y to z.
2. Every $x \in X$ belongs to $P(x)$, so $X = \bigcup_{x \in X} P(x)$. To show that the union is disjoint, it is enough to show that, if $x \in P(y) \cap P(z)$ then $P(y) = P(z)$. Indeed, there is a path from y to x and a path from x to z, so there is a path from y to z. Then, every $w \in P(z)$ can be connected to y by some path, and this means that $P(z) \subseteq P(y)$. Swapping y and z in the previous argument yields that $P(y) \subseteq P(z)$, so $P(y) = P(z)$ as they contain each other.
3. If A is path connected, then either A is empty (and is therefore contained in every path component), or there exists $x \in A$, and every other $y \in A$ can be connected to x by some path. In this case $A \subseteq P(x)$, by definition of $P(x)$.
4. Given any $x \in P(y)$, there is a continuous path from x to y, so its image under f is a continuous path from $f(x)$ to $f(y)$ (this is because a composition of continuous functions is continuous). Therefore $f(x) \in P(f(y))$, and as x was any element of $P(y)$ we get that $f(P(y)) \subseteq P(f(y))$.

Exercise 34. Let $x, y \in \mathbb{Q}$ be distinct points. If we show that there exist two disjoint open subsets $A, B \subset \mathbb{Q}$ such that $x \in A$, $y \in B$, and $A \sqcup B = \mathbb{Q}$, then y cannot belong to $C(x)$, as otherwise $A \cap C(x)$ and $B \cap C(x)$ would be two non-empty, disjoint, open subsets whose union is the connected component $C(x)$. Then it is enough to choose any irrational number $z \in \mathbb{R} - \mathbb{Q}$ such that $x < z < y$ (which clearly exists: prove it!), and set $A = (-\infty, z) \cap \mathbb{Q}$ and $B = (z, +\infty) \cap \mathbb{Q}$.

Exercise 35. Let $A \subseteq \mathbb{R}^n$ be open, and let $x \in A$. For every $\varepsilon > 0$, let

$$B(x, \varepsilon) = \{y \in \mathbb{R}^n \mid |x - y|, \varepsilon\}$$

be the open ball centered at x and with radius ε. As A is open, one can choose ε tiny enough that $B(x, \varepsilon) \subseteq A$. Then $B(x, \varepsilon)$ is a neighbourhood of x (because it is itself open), and we claim that it is path-connected.
To see this, let $y \in B(x, \varepsilon)$, and let $f\colon [0,1] \to \mathbb{R}^n$ map $t \in [0,1]$ to $f(t) = tx + (1-t)y$. In other words, the image of f is the line segment with endpoints x and y. Now, $f(t) \in B(x, \varepsilon)$ for all $t \in [0,1]$. Indeed

$$|x - (tx + (1-t)y)| = |(1-t)(x-y)|,$$

and as $0 \leqslant t \leqslant 1$ we get that

$$|(1-t)(x-y)| = (1-t)|x-y| \leqslant |x-y| < \varepsilon.$$

Then $f([0,1]) \subseteq B(x, \varepsilon)$, that is, f is a path in the ball connecting x to y. As y was any point in $B(x, \varepsilon)$, we get that the ball is path-connected, as required.

Exercise 36. Firstly, we notice that a singleton $\{x\}$ is always closed in a metric space. Indeed, for every $y \neq x$, the open ball $B(y, d(x, y)/2)$ is totally contained in $X - \{x\}$, so the latter is open.
Now, if x is an isolated point then $\{x\}$ is also open. This means that the connected component $C(x)$ cannot contain any other $y \in X$, because otherwise it would be the disjoint union of the two non-empty open subsets $\{x\}$ and $C(x) - \{x\} = C(x) \cap (X - \{x\})$.

Exercise 37. By definition, $x_n \to x$ means that, for every open set A containing x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in A$ for every $n \geqslant n_0$. If by contradiction x was an isolated point, we could choose $A = \{x\}$. But then some x_n should be equal to x, against the hypothesis. □

Chapter 5: Compact spaces

Exercise 38. Let X be a finite topological space, and let $\{A_i\}_{i \in I}$ be an open cover. As X is finite, it has finitely many subsets, so $\{A_i\}_{i \in I}$ must be a finite collection. Hence every open cover has a finite subcover, so X is compact. □

Exercise 39. We first prove that $\mathbb{R}$ is not compact. Indeed, let $A_n = (-n, n)$. The collection $\{A_n\}_{n \in \mathbb{N}_{>0}}$ is an open covering, but it does not admit a finite subcover because if one takes the union of the first k elements one gets

$$\bigcup_{n=1}^{k} A_n = (-k, k).$$

As $\mathbb{R}$ and $(0, 1)$ are homeomorphic, Exercise 40 tells us that $(0, 1)$ is not compact as well.[7]

Exercise 40. Let $f \colon X \to Y$ be a surjective continuous map, and let X be compact. To see that Y is compact as well, let $\{A_i\}_{i \in I}$ be an open covering of Y, and we want to extract a finite subcover. By continuity of f, $\{f^{-1}(A_i)\}_{i \in I}$ is an open covering of X, and as the latter is compact we can extract a finite subcover $\{f^{-1}(A_1) \ldots, f^{-1}(A_n)\}$. In other words, for every $x \in X$, $f(x)$ belongs to at least one between $\{A_1, \ldots, A_n\}$. As f is surjective, every $y \in Y$ is the image of some x, and therefore belongs to at least one between $\{A_1, \ldots, A_n\}$. Thus we proved that $\{A_1, \ldots, A_n\}$ is a finite subcover of $\{A_i\}_{i \in I}$, as required. □

Exercise 41. Let $X = [0, 1]$ and $Y = (0, 1)$, both equipped with the topology inherited from being a subspace of $\mathbb{R}$.[8] Then X is compact by Proposition 5.4, while $(0, 1)$ is not by Exercise 39.

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[7] In order to show that $(0, 1)$ is not compact, one could also emulate the proof for $\mathbb{R}$: try it!

[8] One should check that the subspace topology on Y, *seen as a subspace of* X, is the same as the subspace topology on Y, *seen as a subspace of* $\mathbb{R}$. In general, the same holds for every three topological spaces $Y \subseteq X \subseteq W$. This is an easy but boring exercise, so we skip it.

Exercise 42. Caveat: we assume that both X and Y are non-empty, as $\varnothing \times X = \varnothing$ is compact for every topological space X.
Consider the projection map $\pi_X \colon X \times Y \to X$, mapping a pair (x, y) to its first entry x. This map is clearly surjective; moreover it is continuous, since if $A \subseteq X$ is open then $\pi_X^{-1}(A) = A \times Y$ is a product of open sets, and is therefore open in the product topology. Then by Exercise 40, if $X \times Y$ is compact then X is compact. The same argument holds if one replaces X by Y.

Exercise 43. We want to show that $Z - \overline{B}_R(z)$ is open. To this extent, it is enough to notice that, given any $y \in Z - \overline{B}_R(z)$, the open ball centered at y and with radius $(R - \mathrm{d}_Z(z, y))/2$ is contained in $Z - \overline{B}_R(z)$. This is exactly what we did in Exercise 36 to show that a singleton (that is, a closed ball of radius zero) is closed in a metric space. □

Exercise 44. Consider the open subsets $O_k = Z - A_k$. If by contradiction $\bigcap_{k \in \mathbb{N}} A_k = \varnothing$, then by taking the complements one gets that $\bigcup_{k \in \mathbb{N}} O_k = Z$, that is, $\{O_k\}_{k \in \mathbb{N}}$ is an open covering. As Z is compact, we can extract a finite subcovering, that is, there exists some $r \in \mathbb{N}$ such that $Z = \bigcup_{k \leqslant r} O_k$. However, since $A_{k+1} \subseteq A_k$ for every k, we have that $O_k \subseteq O_{k+1}$. In particular, we have that

$$Z = \bigcup_{k \leqslant r} O_k = O_r.$$

Thus $A_r = Z - O_r = \varnothing$, violating the assumption that the A_k are all non-empty. □

We now provide a possible counterexample for the case when Z is not compact. Let $Z = \mathbb{R}$ with the standard topology, and let $A_k = [k, +\infty)$, which is a decreasing chain of non-empty closed subsets whose intersection is empty.

Exercise 45. Let (X, d) be totally bounded, If one sets $r = 1$, there exist finitely many points $x_1, \ldots x_r$ such that $X \subseteq \bigcup_{i=1}^{r} B_1(x_i)$. Now let $R = \max_{1 \leqslant i,j \leqslant r} \mathrm{d}(x_i, x_j)$, and we claim that, for every two $y, z \in X$ the distance $\mathrm{d}(y, z)$ is at most $R + 2$, thus proving that X is bounded. Indeed, since X is covered by the union of balls, there exist i, j such that $y \in B_1(x_i)$ and $z \in B_1(x_j)$. Then the triangle inequality yields that

$$\mathrm{d}(y, z) \leqslant \mathrm{d}(y, x_i) + \mathrm{d}(x_i, x_j) + \mathrm{d}(x_j, z) \leqslant 1 + R + 1 = R + 2.$$

as required. □

We now provide a possible example of a bounded, but not totally bounded space. Let $X = \mathbb{N}$, and defined a new distance D such that $D(i, j) = 1$ whenever $i \neq j$.[9] Clearly (X, D) is bounded, as any two points are at distance 1. However, if $r < 1$ and $x \in X$, then the ball $B_r(x)$ only contains x, so one cannot cover the infinite set X with finitely many balls of radius r.

Exercise 46. Let $\mathcal{B}$ be a countable base, let $x \in X$ and let $\mathcal{A} = \{B \in \mathcal{B} \mid x \in B\}$, which is countable as $\mathcal{B}$ is. Any open neighbourhood O of x is a union of the elements of the base, so there must be some $B \in \mathcal{B}$ such that $x \in B \subseteq O$. This proves that $\mathcal{A}$ is a countable neighbourhood basis for x, and as $x \in X$ was arbitrary we get that X is first-countable. □

[9] This is the distance inducing the discrete topology!

Exercise 47 (hint). Let $\mathcal{B} = \{(p, q) \mid p < q,\ p, q \in \mathbb{Q}\}$. This is a countable set, as the endpoints belong to $\mathbb{Q}$, and it is easily seen that $\mathcal{B}$ is the basis of *some* topology $\mathcal{O}$ (by checking the requirements of Proposition 3.10).
We now show that $\mathcal{O}$ is actually the standard topology. To do this, one can invoke Exercise 12: if $\mathcal{B}' = \{(a, b) \mid a < b,\ a, b \in \mathbb{R}\}$ is the basis of the standard topology, we must check that:

- For every $B \in \mathcal{B}$ and $x \in B$ there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. To see this, it is enough to notice that $\mathcal{B} \subset \mathcal{B}'$, so one can choose $B' = B$.
- For every $B' \in \mathcal{B}'$ and $x \in B'$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B'$. To see this, suppose that $B' = (a, b)$. Then one can find $p, q \in \mathbb{Q}$ such that $a < p < x < q < b$, so that, if one sets $B \coloneqq (p, q)$, then $x \in B \subseteq B'$.

Thus $\mathcal{O}$ coincides with the standard topology, and therefore $\mathcal{B}$ is a countable basis. $\square$

Exercise 48. Let $\mathcal{B}$ be a countable basis for X, and let $Y \subseteq X$. Then $\mathcal{A} = \{Y \cap B \mid B \in \mathcal{B}\}$ is a countable collection of open sets. To see that $\mathcal{A}$ is a basis, basis, pick open set of Y, which is of the form $Y \cap O$, where O is open in X. Then, as O is a union of elements of $\mathcal{B}$, we get that

$$Y \cap O = Y \cap \left(\bigcup_{i \in I} B_i \right) = \bigcup_{i \in I} B_i \cap Y.$$

Hence $Y \cap O$ is a union of elements of $\mathcal{A}$, i.e. $\mathcal{A}$ is a basis. $\square$

Exercise 49. Let (X, d) be a sequentially compact metric space, and let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence, which we want to show has a limit. By sequential compactness, there exists a subsequence $(x_n(k))_{k \in \mathbb{N}}$ converging to some $\overline{x} \in X$.
We now claim that $\lim_{n \to \infty} x_n = \overline{x}$, by using the definition: for every $\varepsilon > 0$ we want to find $n_0 \in \mathbb{N}$ such that $\mathrm{d}(x_n, \overline{x}) \leqslant \varepsilon$ whenever $n \geqslant n_0$. To see this, let n_0 be such that, whenever $n, m \geqslant n_0$, $\mathrm{d}(x_n, x_m) \leqslant \varepsilon/2$ (such an n_0 exists by definition of a Cauchy sequence). Moreover, choose k_0 such that, whenever $k \geqslant k_0$, $\mathrm{d}(x_{n(k)}, \overline{x}) \leqslant \varepsilon/$ (such a k_0 exists by definition of limit of a sequence). Up to choosing a bigger k_0, we can assume that $n(k_0) \geqslant n_0$. Then, for every $n \geqslant n_0$, the triangle inequality yields

$$\mathrm{d}(x_n, \overline{x}) \leqslant \mathrm{d}(x_n, x_{n(k_0)}) + \mathrm{d}(x_{n(k_0)}, \overline{x}) \leqslant \varepsilon/2 + \varepsilon/2 \leqslant \varepsilon,$$

as required. $\square$

Exercise 50. Let Y be a closed subspace of the complete metric space (X, d), let $\{y_n\}_{n \in \ltimes}$ be a Cauchy sequence in Y, and we want to show that y_n converges to a point in Y. By completeness of X, y_n converges to some $\overline{x} \in X$. Moreover, as a metric space is first-countable (see Remark 3.18), Lemma 3.20 implies that a limit of a sequence in the closed subspace Y must belong to Y, so $\overline{x} \in Y$, as required. $\square$

Exercise 51. Let $x \neq y$ be two points in the metric space (X, d), and let $R = \mathrm{d}(x, y)$. Then the open balls $B(x, R/3)$ and $B(y, R/3)$ are disjoint open subsets, each of which contains only one of the two points. This shows that the topology on X is Hausdorff. $\square$

Exercise 52. For every $x \in D$, Lemma 3.20 grants the existence of two disjoint open sets U_x, V_x such that $C \subseteq U_x$ and $x \in V_x$. Then $\{V_x\}_{x\in D}$ is an open covering of D, from which one can extract a finite covering $\{V_{x_1}, \dots, V_{x_n}\}$. Let $V = V_{x_1} \cup \dots \cup V_{x_n}$, which contains D by definition of a covering, and let $U = U_{x_1} \cap \dots \cap U_{x_n}$. Notice that $C \subseteq U$, as it is contained in every U_{x_i}, and U is open, since it is a finite intersection of open sets. Moreover U and V are disjoint, since for every $i = 1, \dots, n$, $U \subseteq U_{x_i}$ is disjoint from V_{x_i} by assumption. Then U and V satisfy the requirements. □

Chapter 6: The quotient topology

Exercise 53. Firstly, both $X/\sim$ and $\varnothing$ belong to the topology, as their preimages are respectively X and $\varnothing$. Furthermore, given a collection $\{A_i\}_{i\in I}$ of open sets in the quotient topology, we have that

$$\pi^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \pi^{-1}(A_i),$$

and the latter is open in X; hence an arbitrary union of open sets in the quotient topology is open. Finally, given a finite collection $\{A_i\}_{i=1,\dots,n}$ of open sets in the quotient topology, then

$$\pi^{-1}\left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} \pi^{-1}(A_i),$$

and again the latter is open in X; this proves that any finite intersection of open sets in the quotient topology is open.